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Path-Integral Aspects of Supersymmetric
Quantum Mechanics

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Abstract

In this talk we briefly review the concept of supersymmetric quantum mechanics using a model introduced by Witten. A quasi-classical path-integral evaluation for this model is performed, leading to a so-called supersymmetric quasi-classical quantization condition. Properties of this quantization condition are compared with those derived from the standard WKB approach.

1 Introduction

In 1976 Nicolai [1] introduced supersymmetric (SUSY) quantum mechanics as the non-relativistic version of supersymmetric quantum field theory in order to investigate some possible applications of SUSY to spin systems. Independently, in 1981 Witten [2] considered SUSY quantum mechanics as a toy model for studying the SUSY-breaking mechanism in quantum field theory. During the last 15 years SUSY quantum mechanics became an important tool in various branches of theoretical physics. For an overview see, for example, the forthcoming monograph [3].

In this lecture we will begin with a short review on the concept of SUSY in quantum mechanics on the basis of Witten's model. Then we consider a modified stationary-phase approach to the path-integral formulation of this model leading to a supersymmetric version of the van Vleck-Pauli-Morette and Gutzwiller formula for the approximate propagator and Green function, respectively. From the poles of the approximate Green function a quasi-classical supersymmetric quantization condition is derived and compared with the standard WKB condition.

2 SUSY quantum mechanics and Witten's model

Following Nicolai [1] we call a quantum mechanical system characterized by a self-adjoint Hamiltonian H , acting on some Hilbert space \mathcal{H} , *supersymmetric* if there

SUSY is said to be broken. For good SUSY the ground state of H either belongs to H_+ or H_- and is given by

$$\varphi_0^\pm(x) = \varphi_0^\pm(0) \exp \left\{ \pm \int_0^x dz \Phi(z) \right\} \quad (8)$$

Obviously, depending on the asymptotic behavior of the SUSY potential one of the two functions φ_0^\pm will be normalizable (good SUSY) or both are not normalizable (broken SUSY). To be more explicit, let us introduce the *Witten index*, which (according to the Atiyah-Singer index theorem) depends only on the asymptotic values of Φ :

$$\Delta := \text{ind } b = \dim \ker H_- - \dim \ker H_+ = \frac{1}{2} [\text{sgn } \Phi(+\infty) - \text{sgn } \Phi(-\infty)] \quad (9)$$

Hence, for good SUSY we have $\Delta = \pm 1$ with the ground state belonging to H_\mp . For broken SUSY we have $\Delta = 0$. Due to SUSY it is also easy to show [3] that H_+ and H_- are essentially iso-spectral, that is, their strictly positive eigenvalues are identical. These spectral properties of H_\pm are summarized in the following table,

$$\begin{array}{l} \Delta = +1 : E_n^+ = E_{n+1}^- > 0, \quad E_0^- = 0, \\ \Delta = -1 : E_n^- = E_{n+1}^+ > 0, \quad E_0^+ = 0, \\ \Delta = 0 : E_n^- = E_n^+ > 0, \end{array} \quad (10)$$

where $E_n^\pm, n = 0, 1, 2, \dots$, denotes the ordered set of eigenvalues of H_\pm with $E_n^\pm < E_{n+1}^\pm$. For simplicity, we have assumed purely discrete spectra.

3 Quasi-classical path-integral evaluation

Let us now consider a quasi-classical evaluation of the path integral associated with the propagator of the pair of Hamiltonians (7),

$$K_\pm(b, a; \tau) := \langle b | e^{-(i/\hbar)\tau H_\pm} | a \rangle = \int_{x(0)=a}^{x(\tau)=b} \mathcal{D}x \exp \{ i S_0[x] \mp i \varphi[x] \}, \quad (11)$$

where the so-called *tree action* S_0 and *fermionic phase* φ are given by

$$S_0[x] := \frac{1}{2} \int_0^{\tau\hbar/m} dt [\dot{x}^2(t) - \Phi^2(x(t))], \quad \varphi[x] := \frac{1}{2} \int_0^{\tau\hbar/m} dt \Phi'(x(t)) \quad (12)$$

Usually, in the stationary-phase approximation one considers the quadratic fluctuations about stationary paths of the full action $S_\pm[x] := S_0[x] \mp \varphi[x]$. As a modification of this approach we have suggested [4] to consider the fluctuations about the stationary paths of the tree action. These so-called *quasi-classical paths* [5] are

exists a *supercharge* operator Q obeying the following anticommutation relations:

$$\{Q, Q\} = 0 = \{Q^\dagger, Q^\dagger\}, \quad \{Q, Q^\dagger\} = H \quad (1)$$

An immediate consequence of these relations is the conservation of the supercharge and the non-negativity of the Hamiltonian,

$$[H, Q] = 0 = [H, Q^\dagger], \quad H \geq 0 \quad (2)$$

In 1981 Witten [2] introduced a simple model of supersymmetric quantum mechanics. It is defined on the Hilbert space $\mathcal{H} := L^2(\mathbb{R}) \otimes \mathbb{C}^2$, that is, it characterizes a spin- $\frac{1}{2}$ -like particle (with mass $m > 0$) moving along the one-dimensional Euclidean line \mathbb{R} . In constructing a supersymmetric Hamiltonian on \mathcal{H} let us first introduce a bosonic operator b and a fermionic operator f :

$$\begin{aligned} b : L^2(\mathbb{R}) &\rightarrow L^2(\mathbb{R}), & b &:= \frac{1}{\sqrt{2}} \left(\frac{\partial}{\partial x} + \Phi(x) \right), \\ f : \mathbb{C}^2 &\rightarrow \mathbb{C}^2, & f &:= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \end{aligned} \quad (3)$$

where the *SUSY potential* $\Phi : \mathbb{R} \rightarrow \mathbb{R}$ is assumed to be continuously differentiable. Obviously these operators obey the commutation and anticommutation relations

$$[b, b^\dagger] = \Phi'(x), \quad \{f, f^\dagger\} = 1, \quad (4)$$

and allow us to define a suitable supercharge

$$Q := \frac{\hbar}{\sqrt{m}} b \otimes f^\dagger = \frac{\hbar}{\sqrt{m}} \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}, \quad Q^\dagger = \frac{\hbar}{\sqrt{m}} b^\dagger \otimes f = \frac{\hbar}{\sqrt{m}} \begin{pmatrix} 0 & 0 \\ b^\dagger & 0 \end{pmatrix}, \quad (5)$$

which obeys the required relations $\{Q, Q\} = 0 = \{Q^\dagger, Q^\dagger\}$. Note that Q is a combination of a generalized bosonic annihilation operator and a fermionic creation operator. Finally, we may construct a supersymmetric quantum system by defining the Hamiltonian in such a way that also the second relation in (1) holds,

$$H := \{Q, Q^\dagger\} = \frac{\hbar^2}{m} \begin{pmatrix} bb^\dagger & 0 \\ 0 & b^\dagger b \end{pmatrix} = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix}, \quad (6)$$

with

$$H_\pm := \frac{\hbar^2}{2m} \left[-\frac{\partial^2}{\partial x^2} + \Phi^2(x) \pm \Phi'(x) \right] \geq 0 \quad (7)$$

being standard Schrödinger operators acting on $L^2(\mathbb{R})$.

The supersymmetry (1) of a quantum system is said to be a good symmetry (good SUSY) if the ground-state energy of H vanishes. In the other case, $\inf \text{spec } H > 0$,

denoted by x_{qc} , that is, $\delta S_0[x_{qc}] = 0$. Hence, we approximate the tree action to second order in $\eta(t) := x(t) - x_{qc}(t)$ and consider the fermionic phase only along the quasi-classical path:

$$S_{\pm}[x] \approx S_0[x_{qc}] \mp \varphi[x_{qc}] + \frac{1}{2} \int_0^{\tau \hbar/m} dt \left[\dot{\eta}^2(t) + \frac{1}{2} \Phi^2(x_{qc}(t)) \eta^2(t) \right]. \quad (13)$$

Performing the resulting Fresnel-type path integral we arrive at a supersymmetric version of the vanVleck-Pauli-Morette formula [6],

$$K_{\pm}(b, a; \tau) \approx \sum_{x_{qc}}^{\tau \text{ fixed}} \sqrt{\frac{i}{2\pi} \left| \frac{\partial^2 S_0[x_{qc}]}{\partial a \partial b} \right|} \exp \left\{ i S_0[x_{qc}] - i \frac{\pi}{2} \mu[x_{qc}] \mp i \varphi[x_{qc}] \right\}, \quad (14)$$

where $\mu[x_{qc}]$ denotes the Morse index and equals the number of conjugated points along x_{qc} .

In order to obtain some spectral information we pass over from the propagator to the Green function

$$\begin{aligned} G_{\pm}(b, a; \varepsilon) &:= \left\langle b \left| \frac{1}{H_{\pm} - \frac{\hbar^2}{2m} \varepsilon} \right| a \right\rangle \\ &= \frac{1}{i\hbar} \int_0^{\infty} d\tau K_{\pm}(b, a; \tau) \exp\{i\tau \varepsilon \hbar / 2m\}, \quad \text{Im } \varepsilon > 0. \end{aligned} \quad (15)$$

A stationary-phase evaluation of this integral leads to the supersymmetric version of Gutzwiller's formula [6]:

$$\begin{aligned} G_{\pm}(b, a; \varepsilon) &\approx \frac{m}{i\hbar^2} \left| (\varepsilon - \Phi^2(a)) (\varepsilon - \Phi^2(b)) \right|^{-1/4} \\ &\times \sum_{x_{qc}}^{\varepsilon \text{ fixed}} \exp \left\{ i W_0[x_{qc}] - i \frac{\pi}{2} \nu[x_{qc}] \mp i \varphi[x_{qc}] \right\}. \end{aligned} \quad (16)$$

Here $W_0[x_{qc}] := \int_{x_{qc}} dx \sqrt{\varepsilon - \Phi^2(x)}$ denotes Hamilton's characteristic function (associated with the tree action $S_0[x_{qc}]$) and $\nu[x_{qc}]$ is the Maslov index, which equals the number of turning points along x_{qc} . For a single-well shape of Φ^2 the sum in (16) can explicitly be performed [4, 3]. The poles of the resulting formula give rise to the quasi-classical supersymmetric (qc-SUSY) quantization condition

$$\int_{x_L}^{x_R} dx \sqrt{\varepsilon - \Phi^2(x)} = \pi \left(n + \frac{1}{2} \pm \frac{\Delta}{2} \right), \quad (17)$$

where $x_{L/R}$ denote the left and right turning points of x_{qc} , $\Phi^2(x_{L/R}) = \varepsilon = E/\frac{\hbar^2}{2m}$. This quantization condition may be compared with the usual WKB condition de-

rived from a stationary-phase approximation of the full action S_{\pm} ,

$$\int_{x_L}^{x_R} dx \sqrt{\varepsilon - \Phi^2(x)} \mp \Phi'(x) = \pi \left(n + \frac{1}{2} \right), \quad (18)$$

with $q_{L/R}^{\pm}$ as the classical turning points, $\Phi^2(q_{L/R}^{\pm}) \mp \Phi'(q_{L/R}^{\pm}) = \varepsilon = E/\frac{\hbar^2}{2m}$.

4 Discussion

It should be emphasized that the eigenvalues obtained from the qc-SUSY approximation (17) respect all of the spectral properties given in (10). On the contrary, this is in general not the case for the WKB spectrum (18). Furthermore, (17) leads to the exact bound-state spectrum for all so-called shape-invariant potentials. These shape-invariant potentials are known to give rise to factorizable Hamiltonians and hence the eigenvalue problem is easily solvable via the well-known factorization method [7] or an explicit path integral evaluation [8]. This exactness can also be achieved via the WKB approximation, which, however, requires ad hoc Langer-type modifications. Here the question naturally arises: Why is the qc-SUSY approximation exact for those shape-invariant potentials? One possible explanation of this fact via the Nicolai mapping has been discussed in [3]. An alternative explanation may be based on the path-integral generalization of the Duistermaat-Heckman theorem [9].

For various not exactly solvable potentials numerical investigations [6] indicate that for analytical SUSY potentials the qc-SUSY approximation always overestimates the exact energy eigenvalues. Whereas, at least for the case of broken SUSY, the WKB approximation gives an underestimation. For a detailed numerical investigation of this level-ordering phenomenon see the monograph [3], where also other applications of the above quasi-classical approach are discussed.

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References

- [1] H. Nicolai, *Supersymmetry and spin systems*, J. Phys. A **9** (1976) 1497-1506.
- [2] E. Witten, *Dynamical breaking of supersymmetry*, Nucl. Phys. **B188** (1981) 513-554.

- [3] G. Junker, *Supersymmetric Methods in Quantum and Statistical Physics*, Texts and Monographs in Physics, (Springer-Verlag, Berlin, 1996).
- [4] A. Inomata and G. Junker, *Quasi-classical approach to path integrals in supersymmetric quantum mechanics*, in H. Cerdeira, et al., *Lectures on Path Integration: Trieste 1991*, (World Scientific, Singapore, 1993) p.460-482.
- [5] G. Junker and S. Matthiesen, *Supersymmetric classical mechanics*, J. Phys. A **27** (1994) L751-L755; ADDENDUM: *Pseudoclassical mechanics and its solution*, J. Phys. A **28** (1995) 1467-1468.
- [6] A. Inomata and G. Junker, *Quasiclassical path-integral approach to supersymmetric quantum mechanics*, Phys. Rev. A **50** (1994) 3638-3649.
- [7] L. Infeld and T.E. Hull, *The factorization method*, Rev. Mod. Phys. **23** (1951) 21-68.
- [8] A. Inomata in A. Inomata, H. Kuratsuji and C.C. Gerry, *Path Integrals and Coherent States of SU(2) and SU(1,1)*, (World Scientific, Singapore, 1992) p.1-138.
- [9] M.F. Atiyah, *Circular symmetry and stationary phase approximation*, in Proceedings of the conference in honor of L. Schwartz, (Astérisque, Paris, 1984).